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6 RECURSIVE RELATIONS IN THE COMPUTATION OF THE EQUILIBRIUM RESULTS OF FINITE QUEUES.

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ABSTRACT

Imbedded Markov chains of finite queueing stystems with unit jumps at regeneration points have an almost left triangular (in systems of the type G/M/s/N - in Kendall notation modified to include system capacity) or an almost right triangular (in systems of the type M/G/1/N) structure. Using this structure a fundamental recursion on the elements of the transition probability matrix is developed, which in turn helps derive first passage as well as equilibrium results in computationally feasible forms. The computational procedure is illustrated using the system G/M/s/N with two arrival classes and priority service.

1. INTRODUCTION

Finite queueing systems have not been studied so far as extensively as those with countably or uncountably infinite state space. The main reasons for this situation seem to be the suitability of an infinite state system approximation for a finite state system with a moderately large state space and the relative simplicity of a finite queueing system with a very small state space. However, when the state space is of medium size, in order to provide usable results, the study of finite queues by themselves is necessary. In this paper we present an analytic technique which should prove significant in a wide variety of such cases.

The class of queueing systems that can be analyzed by the methods developed in this paper may be identified through two subclasses.

(1) Multiserver queueing systems with a general arrival process (possibly dependent on the state of the system), exponential service times and a waiting room of finite capacity (denoted as G/M/s/N).

(2) Single server queueing systems, with Poisson arrivals (possibly dependent on the state of the system) and a waiting room of finite capacity (denoted as M/G/1/N).

The procedure is based on the imbedded Markov chain of the basic process and therefore it is essential that in the case of multi-server systems the service times be exponential and in single server systems when the service times have a general distribution the arrival process be Poisson. It is to be noted, in queueing systems belonging to the above classes, the transition probability matrices (TPM) of the imbedded chains have an almost triangular structure (also known as Hessenberg matrices). If arrivals are one at a time in the first subclass of queues, the TPM is almost left triangular (a matrix in which all the elements above the super diagonal are zero) and when the departures are

one at a time in the second subclass of systems, the TPM is almost right triangular (a matrix in which all the elements below the sub-diagonal are zero).

In the next two sections we consider an almost left triangular transition probability matrix and an almost right triangular transition probability matrix respectively. Each section addresses to the problem of deriving the fundamental recursion, limiting distribution and first passage times. Remarks have also been incorporated on the computational feasibility of the results presented in the paper. Section 4 on computational procedure outlines the algorithmic aspects of the model. The final section illustrates this procedure with an example, where a system of the type G/M/s/N described above is analyzed to give most of the measures of performance of interest. Such systems arise in the study of telecommunications traffic at a multiple channel node shared by more than one class of customers (e.g., voice and data; see Fischer [3]; for a complete analysis of the system see, Bhat and Raju [1]).

2. AN ALMOST LEFT TRIANGULAR TPM

Two standard results from the theory of finite Markov chains are given below.

(1) Let $\{0,1,2,\ldots, N\}$ be the state space and P_{ij} $(i,j=0,1,2,\ldots N)$ be the one step transition probabilities of an aperiodic, recurrent and irreducible Markov chain $\{Q_n, n=0,1,2,\ldots\}$. Let $\pi=(\pi_0,\pi_1,\ldots,\pi_N)$ be the limiting distribution of the Markov chain. Then π can be obtained by solving the set of simultaneous linear equations

$$\pi P = \pi \tag{1}$$

using the normalizing condition $\sum_{j=0}^{N} \pi_j = 1$, where P is the transition probability matrix with elements P_{ij} $(i,j=0,1,2\ldots N)$.

(2) Consider the first passage of the Markov chain from a state $k \epsilon B_i$ where $B_i = \{i, i+1, \ldots, N\}$ to any state $\epsilon \bar{B}_i$ where $\bar{B}_i = \{0, 1, 2, \ldots, i-1\}$. Define

$$T_k^N(B_i) = \inf\{n \mid Q_n \epsilon \bar{B}_i, \text{ given } k \epsilon B_i\}; \; T_k^N \equiv T_k^N(B_1). \tag{2}$$

Let H be the substochastic matrix obtained by partitioning the transition probability matrix P as follows:

$$P = \left[\begin{array}{cccc|ccc} P_{00} & P_{01} & \cdots & P_{0,\, i-1} & P_{0i} & \cdots & P_{0N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \hline P_{i0} & P_{i1} & \cdots & P_{i,\, i-1} & & & \\ \vdots & \vdots & & \vdots & & H & \\ P_{N0} & P_{N1} & \cdots & P_{N,\, i-1} & & & \end{array}\right]$$

Then it is known (see Kemeny and Snell [4]) that $E[T_k^N(B_i)]$ is given by the $(k-i+1)$th row sum of $(I-H)^{-1}$ [known as the fundamental matrix] and the elements of this row give the mean number of visits of the Markov chain to the corresponding states before the eventual first passage.

Variance of first passage times can also be determined as functions of elements of $(I-H)^{-1}$.

Given below are results that can be used to simplify the matrix inversion procedure needed in determining the first passage times and the limiting distribution. The method exploits the almost triangular structure of the TPM and is based on recursive relations convenient for computational use.

2.1 The fundamental recursion

Let the transition probability matrix P have an almost left triangular structure. Without loss of generality, for notational convenience we shall assume that the first passage is from the set of states $\{1, 2, \ldots N\}$ to state $\{0\}$. Now, writing $H \equiv H(N)$, where N is the order of H, we have

$$
I-H(N) = \begin{bmatrix}
1-P_{11} & -P_{12} & & & \\
-P_{21} & 1-P_{22} & \cdot & \bigcirc & \\
\vdots & & & & \\
-P_{N-1,1} & -P_{N-1,2} & \cdots & 1-P_{N-1,N-1} & -P_{N-1,N} \\
-P_{N1} & -P_{N2} & \cdots & -P_{N,N-1} & 1-P_{NN}\,.
\end{bmatrix} \tag{3}
$$

For $j>0$, $k\geq 0$, define

$$
A_{k+1}^{(j)} = \begin{bmatrix}
1-P_{jj} & -P_{j+1,j} & \cdots & -P_{j+k-2,j} & -P_{j+k-1,j} & -P_{j+k,j} \\
-P_{j,j+1} & 1-P_{j+1,j+1}, & \cdots & -P_{j+k-2,j+1} & -P_{j+k-1,j+1} & -P_{j+k,j+1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
 & & & -P_{j+k-2,j+k-1} & 1-P_{j+k-1,j+k-1} & -P_{j+k,j+k-1} \\
 & \bigcirc & & & -P_{j+k-1,j+k} & 1-P_{j+k,j+k}
\end{bmatrix} \tag{4}
$$

and let $\det(A_{k+1}^{(j)}) = \Delta_{k+1}^{(j)}$. Using the principal minors of $A_{k+1}^{(j)}$ we have the recurrence relation

$$\Delta_k^{(j)} \equiv 0 \quad \text{for } k<0$$

$$\Delta_0^{(j)} \equiv 1$$

$$
\Delta_{k+1}^{(j)} = (1-P_{j+k,j+k})\Delta_k^{(j)} - P_{j+k-1,j+k}\,P_{j+k,j+k-1}\Delta_{k-1}^{(j)} - \ldots
- \left(\prod_{\ell=0}^{k-1} P_{j+k-1-\ell,j+k-\ell}\right) P_{j+k,j}\Delta_0^{(j)}\,. \tag{5}
$$

(see, Faddeev and Faddeeva [2], p. 31).

For computational purposes a better recursive relation than (5) can be derived. Define

$$\beta_k^{(j)} = \prod_{n=j+1}^{k+j} P_{n-1,n} \qquad (j>0, k\geq 0)$$

$$\beta_0^{(j)} \equiv 1 \quad \text{for all } j>0$$

and $\quad a_k^{(j)} = \Delta_k^{(j)} \Big/ \beta_k^{(j)}\,.$

Dividing (5) by $\beta_{k+1}^{(j)}$ and simplifying we get

$$a_{k+1}^{(j)} = \frac{1}{P_{j+k,j+k+1}}\left[a_k^{(j)} - \sum_{\ell=0}^{k} P_{j+k,j+k-\ell}\, a_{k-\ell}^{(j)}\right] (j\geq 1,\ 0\leq k\leq N-j-1) \qquad (6)$$

where for $j > 0$,

$$a_0^{(j)} \equiv 1$$

$$a_1^{(j)} = (1-P_{jj}) \Big/ P_{j,j+1}$$

$$a_n^{(j)} \equiv 0 \quad \text{for all } n<0.$$

The recursive relation (6) has the following equivalent form

$$a_{k+1}^{(j)} = \left[\frac{1-P_{jj}}{P_{j,j+1}}\right] a_k^{(j+1)} - \sum_{\ell=1}^{k}\left[\frac{P_{j+\ell,j}}{P_{j+\ell,j+1+\ell}}\right] a_{k-\ell}^{(j+1+\ell)}$$

$$(j\geq 1,\ 0\leq k\leq N-j)\ . \qquad (7)$$

The equivalence of the two forms can be established directly using mathematical induction.

Because of its usefulness in computations we shall identify equation (7) as the <u>fundamental recursion</u> and the successive iterates as <u>transition iterates</u>. The limiting distribution and the first passage times for the TPM can be expressed in terms of these computationally attractive transition iterates.

2.2 Limiting distribution of the Markov chain

In obtaining the limiting distribution the following inversion formula is useful.

Lemma 2.2.1: Let $B(n,k)$ be a matrix given by

$$
B(n,k) = \begin{bmatrix}
-P_{k,k+1} & & & & \\
1-P_{k+1,k+1} & -P_{k+1,k+2} & & & \\
\vdots & & & & \\
-P_{n+k-1,k+1} & -P_{n+k-1,k+2} & \cdots & 1-P_{n+k-1,n+k-1} & -P_{n+k-1,n+k}
\end{bmatrix}.
$$

Then, the inverse of $B(n,k)$ is given by

$$
B(n,k)^{-1} = (-1) \begin{bmatrix}
1/P_{k,k+1} & & & \\
a_1^{(k+1)}/P_{k,k+1} & 1/P_{k+1,k+2} & & \\
a_2^{(k+1)}/P_{k,k+1} & a_1^{(k+2)}/P_{k+1,k+2} & & \\
\vdots & a_2^{(k+2)}/P_{k+1,k+2} & & \\
\vdots & \vdots & & \\
a_{n-1}^{(k+1)}/P_{k,k+1} & a_{n-2}^{(k+2)}/P_{k+1,k+2} & \cdots \; a_1^{(n+k-2)}/P_{n+k-2,n+k-1} & \\
 & & & 1/P_{n+k-1,n+k}
\end{bmatrix} \tag{8}
$$

where $a_k^{(j)}$ are transition iterates defined in (6) or (7).

Proof: It is sufficient to show that for any i ($1 \le i \le n$) the following system of equations is satisfied.

$$
[B^{-1}(n,k)][B(n,k)_{.i}] = I_{.i} \tag{9}
$$

where $B(n,k)_{.i}$ is the i^{th} column of $B(n,k)$ and $I_{.i}$ is the unit column vector with 1 in the i^{th} position. Expanding the equations, we have

$$-\frac{1}{P_{k+i-1,k+i}}\left(-P_{k+i-1,k+i}\right) = 1$$

$$P_{k+i-1,k+i}\,\frac{a_1^{(k+i)}}{P_{k+i-1,k+i}} - \frac{1-P_{k+i,k+i}}{P_{k+i,k+i+1}} = 0$$

and in general for any j $(1 \le j \le n-i-1)$,

$$a_j^{(k+i)} - \left[\frac{a_{j-1}^{(k+i+1)}}{P_{k+i,k+i+1}} - \sum_{\ell=0}^{j-1}\left(\frac{P_{k+i+\ell,k+i}}{P_{k+i+\ell,k+i+\ell+1}}\right) a_{j-1-\ell}^{(k+i+1+\ell)}\right] = 0\ .$$

From the fundamental recursion it is obvious that the above equality holds. This completes the proof of the lemma.

Based on this lemma we have the following theorem.

<u>Theorem 2.2.1</u>: Let $\{Q_n, n=0,1,2,\ldots\}$ be an aperiodic, irreducible finite Markov chain with an almost left triangular transition probability matrix with elements P_{ij} $(i,j=0,1,2,\ldots N)$. Let $\pi=(\pi_0,\pi_1,\ldots,\pi_N)$ be the limiting distribution of the Markov chain. Then the elements of π are given by

$$\pi_j = \left(a_{N-j}^{(j+1)} \Big/ P_{j,j+1}\right)\pi_N \qquad (0 \le j \le N-1) \tag{10}$$

$$\pi_N = \left[1 + \sum_{j=0}^{N-1}\left(a_{N-j}^{(j+1)} \Big/ P_{j,j+1}\right)\right]^{-1}$$

where $a_k^{(j)}$ are the transition iterates defined in (7) and we have assumed $P_{N,N+1} \equiv 1$ to facilitate computation.

<u>Proof</u>: Rewriting (1) we have

$$\pi(I-P) = 0 \tag{11}$$

where I is the identity matrix of rank (N+1). Realizing that the vector π can be uniquely determined from the last N equations of (11) and the nor-

malizing contition $\sum_{j=0}^{N} \pi_j = 1$, we re-arrange the last N equations of (11) as follows

$$[\pi_0, \pi_1, \ldots, \pi_{N-1}] \begin{bmatrix} -P_{01} & & & & \\ 1-P_{11} & -P_{12} & & & \\ -P_{21} & 1-P_{22} & \cdot & & \\ \vdots & & & & \\ -P_{N-1,1} & -P_{N-1,2} & \cdots & 1-P_{N-1,N-1} & -P_{N-1,N} \end{bmatrix}$$

$$= \pi_N\, [P_{N1}, P_{N2}, \ldots, P_{N,N-1}, -(1-P_{NN})].$$

The coefficient matrix of the above non-homogeneous system of linear equations is B(N,0) of lemma 2.2.1. Thus we have

$$[\pi_0, \pi_1, \ldots, \pi_{N-1}] = \pi_N [P_{N1}, P_{N2}, \ldots P_{N,N-1}, -(1-P_{NN})] B^{-1}(N,0).$$

which yields

$$\pi_0 = \frac{\pi_N}{P_{01}} \left[a_{N-1}^{(1)} - \sum_{\ell=0}^{N-1} P_{N,N-\ell}\, a_{N-1-\ell}^{(1)}\right]$$

$$\pi_j = \frac{\pi_N}{P_{j,j+1}} \left[a_{N-1-j}^{(j+1)} - \sum_{\ell=0}^{N-1-j} P_{N,N-\ell}\, a_{N-1-j-\ell}^{(j+1)}\right] \quad 1 \le j \le N-1$$

which can be further simplified by using the recursion (6) and assuming $P_{N,N+1} \equiv 1$ for notational convenience. (It may be easily verified that this assumption does not invalidate the recursion.) The theorem now follows from the normalizing condition $\sum_{j=0}^{N} \pi_j = 1$.

2.3. $(I-H)^{-1}$ and the first passage times.

The objective of this section is to obtain the elements of the inverse matrix $(I-H)^{-1}$ using some of the special structural properties of the almost left triangular TPM. We have

<u>Lemma 2.3.1</u>:

$$\det[I-H] = \Delta_N^{(1)} . \tag{12}$$

This result is obvious from our definitions following equation (4).

<u>Corollary</u>: Let $C(1,1)$ be the co-factor of the (1,1) element of matrix $[I-H]$. Then

$$C(1,1) = \Delta_{N-1}^{(2)} . \tag{13}$$

<u>Lemma 2.3.2</u>: The co-factor $C(N,1)$ of the $(N,1)$ element of matrix $[I-H]$ is given by

$$C(N,1) = \prod_{\ell=1}^{N-1} P_{\ell,\ell+1} . \tag{14}$$

This result follows if we note that $C(N,1)$ is the determinant of a triangular matrix of dimension $(N-1)$ with $-P_{\ell,\ell+1}$ $(1\leq\ell\leq N-1)$ for each of its diagonal entries.

For further development we introduce the following notations. Let D^T be the transpose of matrix D. Define

$$\begin{aligned} X &\equiv (A_N^{(1)})^T = [I-H] \\ Y &\equiv (A_{N-1}^{(2)})^T . \end{aligned} \tag{15}$$

Denote the adjoint matrices of X and Y by

$$\begin{aligned} \mathrm{adj}(X) &= ||x_{kj}|| \quad (1\leq k,j\leq N) \\ \mathrm{adj}(Y) &= ||y_{kj}|| \quad (1\leq k,j\leq N-1). \end{aligned}$$

Then we have

<u>Lemma 2.3.3</u>:

$$x_{1j} = P_{12}y_{1,j-1} \quad (2\leq j\leq N) . \tag{16}$$

<u>Proof</u>: We have the set of equations

$$\mathrm{adj}(X_{1.})X = \det(X)I_1 . \tag{17}$$

where $adj(X_1.)$ is the first row of $adj(X)$ and $I_1.$ is the unit row vector with 1 in the first position. But we already know x_{11} from (13) and x_{1N} from (14). Therefore the remaining N-2 elements of $adj(X_1.)$ can be uniquely determined from the last N-2 equations of (17) by back substitution. Thus dropping the first two equations of (17) we have the linear homogeneous system

$$[x_{12}, x_{13}, \ldots, x_{1N}] \begin{bmatrix} -P_{23} & & & & \\ 1-P_{33} & -P_{34} & & & \\ \vdots & & & & \\ -P_{N-1,3} & -P_{N-1,4} & \ldots & 1-P_{N-1,N-1} & -P_{N-1,N} \\ -P_{N3} & -P_{N4} & \ldots & -P_{N,N-1} & 1-P_{NN} \end{bmatrix}$$

$$= [0,0,\ldots, 0]. \tag{18}$$

Similarly from the set of equations

$$adj(Y_1.)Y = det(Y)I_1.$$

(with definitions similar to those for X), dropping the first equation we get the same system of N-2 equations as (18) except that the variable vector now is $[y_{11}, y_{12}, \ldots, y_{1,N-1}]$. Also the solution to the set of equations (18) is unique if we know one of the elements of the variable vector. Further from (14) we have

$$x_{1N} = P_{12}y_{1,N-1}$$

which relationship should then hold for all elements.

This completes the proof of the lemma. The above lemmas lead us to the following theorem.

Theorem 2.3.1

$$\sum_{j=1}^{N} x_{1j} = \sum_{j=1}^{N-2} \Delta_{N-j}^{(j+1)} \prod_{\ell=2}^{j} P_{\ell-1,\ell} + \prod_{\ell=2}^{N-1} P_{\ell-1,\ell} \, [1-P_{NN} + P_{N-1,N}]. \qquad (19)$$

where the product is assumed to be 1 whenever the lower limt is larger than the upper limit.

Proof: First note

$$\sum_{j=1}^{N} x_{1j} = \sum_{j=1}^{N-2} x_{1j} + x_{1N-1} + x_{1N}.$$

But from (14) we have

$$x_{1N} = \prod_{\ell=2}^{N} P_{\ell-1,\ell}$$

and the last equation in (18) gives

$$(1-P_{NN})x_{1N} = P_{N-1,N}x_{1,N-1}$$

from which we get

$$x_{1,N-1} = (1-P_{NN}) \prod_{\ell=2}^{N-1} P_{\ell-1,\ell}.$$

Hence

$$x_{1,N-1} + x_{1N} = [1-P_{NN} + P_{N-1,N}] \prod_{\ell=2}^{N-1} P_{\ell-1,\ell}. \qquad (20)$$

Also from (13) we have

$$x_{11} = \Delta_{N-1}^{(2)} .$$

The remaining terms can be generated recursively as follows by repeatedly applying (16) and (13).

$$
\begin{aligned}
x_{12} &= P_{12}[C(1,1) \text{ of } (A_{N-1}^{(2)})^T] \\
&= P_{12}\Delta_{N-2}^{(3)} \\
x_{13} &= P_{12}[C(1,2) \text{ of } (A_{N-1}^{(2)})^T] \\
&= P_{12} \cdot P_{23}[C(1,1) \text{ of } (A_{N-2}^{(3)})^T] \\
&= P_{12} \cdot P_{23}\Delta_{N-3}^{(4)} \\
&\vdots \\
x_{1j} &= \Delta_{N-j}^{(j+1)} \prod_{\ell=2}^{j} P_{\ell-1,\ell}
\end{aligned}
$$

which completes the proof of the theorem.

Theorem 2.3.1 provides essentially the needed results to give the mean first passage time (in terms of number of transitions) $E[T_1^N]$. We have

Theorem 2.3.2: Let $\{Q_n, n=0,1,2... \}$ be an aperiodic, irreducible finite Markov chain with an almost left triangular transition probability matrix with elements P_{ij} $(i,j=0,1,2,..., N)$. Let T_1^N be the first passage time from state 1 to state 0, in terms of number of transitions. Then

$$
E[T_1^N] = \frac{\left(\dfrac{1-P_{NN}+P_{N-1,N}}{P_{N-1,N}}\right) + \sum_{j=1}^{N-2} \dfrac{1}{P_{j,j+1}}\left(a_{N-1-j}^{(j+1)} - \sum_{\ell=0}^{N-1-j} P_{N,N-\ell}\, a_{N-1-j-\ell}^{(j+1)}\right)}{a_{N-1}^{(1)} - \sum_{\ell=0}^{N-1} P_{N,N-\ell}\, a_{N-1-\ell}^{(1)}} . \tag{21}
$$

Alternately defining $P_{N,N+1} \equiv 1$ to facilitate computations

$$
E[T_1^N] = \left[1 + \sum_{j=1}^{N-1} \frac{a_{N-j}^{(j+1)}}{P_{j,j+1}} \right] \Bigg/ a_N^{(1)} . \tag{22}
$$

Proof: Based on discussion following (2) we have

$$
E[T_1^N] = \frac{\text{First row sum of adj}[I-H(N)]}{\det[I-H(N)]} .
$$

Substituting from (19) and (12) we get

$$E[T_1^N] = \sum_{j=1}^{N-2} \Delta_{N-j}^{(j+1)} \prod_{\ell=2}^{j} P_{\ell-1,\ell} + \prod_{\ell=2}^{N-1} P_{\ell-1,\ell}\left(1-P_{NN} + P_{N-1,N}\right)\Bigg/\Delta_N^{(1)} .$$

Dividing the numerator and denominator by $\prod_{\ell=2}^{N} P_{\ell-1,}$ we get

$$E[T_1^N] = \left[\left(\frac{1-P_{NN}-P_{N-1,N}}{P_{N-1,N}}\right) + \sum_{j=1}^{N-2} \frac{\Delta_{N-j}^{(j+1)}}{\prod_{\ell=j+1}^{N} P_{\ell-1,\ell}}\right]\Bigg/\left(\frac{\Delta_N^{(1)}}{\prod_{\ell=2}^{N} P_{\ell-1,\ell}}\right). \tag{23}$$

However from equation (6) and noting the following definition, it follows that

$$\frac{\Delta_{N-j}^{(j+1)}}{\prod_{\ell=j+1}^{N} P_{\ell-1,\ell}} = \frac{1}{P_{j,j+1}}\left(a_{N-1-j}^{(j+1)} - \sum_{\ell=0}^{N-1-j} P_{N,N-\ell} A_{N-1-j-\ell}^{(j+1)}\right) \tag{24}$$

and

$$\frac{\Delta_N^{(1)}}{\prod_{\ell=2}^{N} P_{\ell-1,\ell}} = a_{N-1}^{(1)} - \sum_{\ell=0}^{N-1} P_{N,N-\ell}\, a_{N-1-\ell}^{(1)} . \tag{25}$$

Now, substituting from (24) and (25) in (23) we get the required expression (21).

In the recursive relation (6) assuming k=N-1 (note that the recursion restricts k to be less than N-1) we get

$$a_N^{(1)} = \frac{1}{P_{N,N+1}}\left[a_{N-1}^{(1)} - \sum_{\ell=0}^{N-1} P_{N,N-\ell}\, a_{N-1-\ell}^{(1)}\right] . \tag{26}$$

Since the Markov chain is assumed to have N states $P_{N,N+1}$ does not exist. However, if we assume $P_{N,N+1} \equiv 1$, the transition iterates $a_j^{(1)}$ $(0 \leq j \leq N-1)$ remain unchanged and thus $a_N^{(1)}$ in this case actually represents the term within the braces of the right hand side of equation (26). Similarly, we can show that

$$a_{N-j}^{(j+1)} = \frac{1}{P_{N,N+1}}\left[a_{N-1-j}^{(j+1)} - \sum_{\ell=0}^{N-1-j} P_{N,N-\ell}\, a_{N-1-j-\ell}^{(j+1)}\right] . \tag{27}$$

Now, substituting (26) and (27) in (23) and making a change of variable we obtain the required expression (22). This completes the proof of the theorem.

It may be recalled that, initially without loss of generality in the procedure, we considered the first passage from state 1 to state 0. On the other hand, if the taboo states had been {0,1,2...i-1} and the first passage considered was from state i to any one of these states, the following expressions would have resulted instead of (21) and (22).

$$E[T_i^N(B_i)]$$

$$= \frac{\left(\dfrac{1-P_{NN}+P_{N-1,N}}{P_{N-1,N}}\right) + \sum_{j=1}^{N-i-1} \dfrac{1}{P_{i+j-1,i+j}} \left(a_{N-i-j}^{(i+j)} - \sum_{\ell=0}^{N-i-j} P_{N,N-\ell}\, a_{N-i-j-\ell}^{(i+j)}\right)}{a_{N-i}^{(i)} - \sum_{\ell=0}^{N-i} P_{N,N-\ell}\, a_{N-i-\ell}^{(i)}} \tag{28}$$

or

$$= \left[1 + \sum_{j=1}^{N-i} \frac{a_{N-i+1-j}^{(i+j)}}{P_{i+j-1,i+j}}\right] \Bigg/ a_{N-i+1}^{(i)} \tag{29}$$

with $P_{N,N+1} \equiv 1$.

So far we have concentrated on the first row of the matrix $(I-H)^{-1}$ yielding the mean first passage time from state 1 to state zero. To determine first passage times from other states all elements of $(I-H)^{-1}$ need to be known. They are given in the following Lemma.

<u>Lemma 2.3.4</u>

Let adj[I-H] be denoted by

$$X = \| x_{jk} \| \qquad (j,k=1,2,3,\ldots N),$$

Then
$$x_{jk} = \left(\prod_{\ell=2}^{N} P_{\ell-1,\ell}\right) \left[\frac{a_{N-k}^{(k+1)}\, a_{j-1}^{(1)} - a_N^{(1)}\, a_{j-k-1}^{(k+1)}}{P_{k,k+1}}\right] \tag{30}$$

<u>Proof</u>: For the sake of conciseness only essential steps of the proof will be given.

Clearly
$$[I-H].\,X._k = \det[I-H].I._k \tag{31}$$
$$(1 \le k \le N.)$$

From Theorem 2.3.1, x_{1k} ($1\leq k\leq N$) are known. Hence for any k, the remaining N-1 elements of $X_{.k}$ can be uniquely determined from any N-1 equations of the independent non-homogenous system of equation (31). Dropping the last equation of (31) and re-arranging we have for k = 1,2...,N-1

$$B(N-1,1)\begin{bmatrix} x_{2K} \\ x_{3K} \\ \cdot \\ x_{NK} \end{bmatrix} = \det[I-H]I._{k} - x_{1k}\begin{bmatrix} 1-P_{11} \\ -P_{21} \\ \cdot \\ \cdot \\ -P_{-1,1} \end{bmatrix}$$

where B(n,k) is as defined in Lemma 2.2.1, and I is a (N-1)x(N-1) identity matrix. Premultiplying both sides by $B^{-1}(N- ,)$, using the re-cursion (6) as a simplifying feature and the form of the inverse of $B(N-i,i)$ we get for (k=1,2,...,N-1)

$$\begin{bmatrix} x_{2k} \\ x_{3k} \\ \cdot \\ \cdot \\ \cdot \\ x_{NK} \end{bmatrix} = x_{1k}\begin{bmatrix} a_1^{(1)} \\ a_2^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ a_{N-1}^{(1)} \end{bmatrix} - \frac{\det[I-H]}{P_{k,k+1}}\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \\ a_1^{(k+1)} \\ \cdot \\ \cdot \\ \cdot \\ a_{N-1-k}^{(k+1)} \end{bmatrix} \qquad (32)$$

A similar operation for k=N gives

$$\begin{bmatrix} x_{2N} \\ x_{3N} \\ \cdot \\ \cdot \\ \cdot \\ x_{NN} \end{bmatrix} = x_{1N}\begin{bmatrix} a_1^{(1)} \\ a_2^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ a_{N-1}^{(1)} \end{bmatrix} \qquad (33)$$

The lemma now follows by substituting known expressions for det[I-H] and x_{1k} (k=1,2... N).

Corollary

(a) The elements of the matrix $(I-H)^{-1} = \| \nu_{jk} \|$ $(j,k=1,2,\dots N)$ are given by

$$\nu_{jk} = \left[\frac{a_{N-k}^{(k+1)} a_{j-1}^{(1)}}{a_N^{(1)}} - a_{j-k-1}^{(k+1)} \right] \Big/ P_{k,k+1} \quad . \tag{34}$$

(b) If the first passage considered is from a state $\varepsilon B_i = \{i,i+1,\dots N\}$ to a state $\varepsilon \bar{B}_i = \{0,1,2,\dots i-1\}$, we would have considered an H matrix with dimension N-i+1. Then, let $[I-H(N-i+1)]^{-1} = \| \nu_{jk}(N-i+1) \|$ $(j,k=1,2\dots, N-i+1)$. The elements $\nu_{jk}(N-i+1)$ are obtained as

$$\nu_{jk}(N-i+1) = \left[\frac{a_{N-i+1-k}^{(i+k)} a_{j-1}^{(i)}}{a_{N-i+1}^{(i)}} - a_{j-k-1}^{(i+k)} \right] \Big/ P_{i+k-1,i+k}$$

The proof of this result follows exactly the same lines as that of (34).

Theorem 2.3.3: (a) Let $E[T_k^N(B_1)]$ be the first passage time of the Markov chain to state 0, having initially started from state k. Then

$$E[T_k^N(B_1)] = E[T_1^N] a_{k-1}^{(1)} - \sum_{j=1}^{k-1} \frac{a_{k-j-1}^{(j+1)}}{P_{j,j+1}} \quad (k=2,3,\dots N) \tag{35}$$

where

$$E[T_1^N] = \left[1 + \sum_{j=1}^{N-1} \frac{a_{N-j}^{(j+1)}}{P_{j,j+1}} \right] \Big/ a_N^{(1)} .$$

(b) Let $E[T_k^N(B_i)]$ be the mean first passage time of the Markov chain to any of the states $\varepsilon \bar{B}_i = \{0,1,2\dots i-1\}$ from state $k \varepsilon B_i = \{i,i+1,\dots, N\}$. Then

$$E[T_k^N(B_i)] = a_{k-i}^{(i)} \, E[T_i^N(B_i)] - \sum_{m=1}^{k-i} \frac{a_{k-i-m}^{(i+m)}}{P_{i+m-1,i+m}}$$

where

$$E[T_i^N(B_i)] = \left[1 + \sum_{m=1}^{N-i} \frac{a_{N-i+1-m}^{(i+m)}}{P_{i+m-1,i+m}} \right] \Big/ a_{N-i+1}^{(i)} .$$

Part (a) of the theorem follows from direct substitution. Part (b) can be derived by similar methods.

2.4 Remarks on computational feasibility

A comparison of expressions for the limiting distribution π and the first passage time $E[T_1^N]$ indicates that, if we compute π, we have all the information needed to obtain the latter. To determine the steady state probability vector π, we need only N transition iterates $a_{N-j}^{(j+1)}$, $(j=0,1,\ldots,N-1)$. Noting that $a_1^{(N)} = (1-P_{NN})/P_{N,N+1}$ where $P_{N,N+1}=1$ the remaining N-1 transition iterates can be generated successively using the fundamental recursion (7), and starting with $k=1, j=N-k$ and successively incrementing k up to N-1. Given the matrix P, it can be deduced that the generation of N successive transition iterates would require a total $N^2+\frac{N(N+1)}{2}$ operations (N^2- multiplications/divisions and $\frac{N(N+1)}{2}$ - subtractions/additions) on a digital computer.

Many times, systems such as G/M/s/N result in a transition probability matrix with last two rows identical. Modified expressions for mean first passage times and the limiting distribution can be given for this case much the same way as the general case. However, the resulting expressions are computationally advantageous only in systems with small $N (N \leq 6)$ or in systems such as G/M/1/N in which the recursions (6) and (7) turn out to be the same.

3. AN ALMOST RIGHT TRIANGULAR TPM

The procedure for analysis of an almost right triangular TPM would follow easily if we note the following.

For a vector $X = (x_1, x_2, x_3, \ldots, x_{n-1}, x_n)$, define its reverse to be:

$$X^* = (x_n, x_{n-1}, \ldots, x_3, x_2, x_1).$$

For a matrix

$$Y = \begin{bmatrix} y_{11} & y_{12} & & y_{1,n-1} & y_{1n} \\ y_{21} & y_{22} & & y_{2,n-1} & y_{2n} \\ & & & & \\ & & & & \\ y_{n-1,1} & y_{n-1,2} & & y_{n-1,n-1} & y_{n-1,n} \\ y_{n1} & y_{n2} & & y_{n,n-1} & y_{n,n} \end{bmatrix}$$

define its reverse to be

$$Y^* = \begin{bmatrix} y_{nn} & y_{n,n-1} & & y_{n2} & y_{n1} \\ y_{n-1,n} & y_{n-1,n-1} & & y_{n-1,2} & y_{n-1,1} \\ & & & & \\ & & & & \\ y_{2n} & y_{2,n-1} & & y_{22} & y_{21} \\ y_{1n} & y_{1,n-1} & & y_{12} & y_{11} \end{bmatrix}$$

Then,

i) $XY = X$ and $X^*Y^* = X^*$ represent identical sets of linear equations

ii) If Y is ALT, then Y^* is ART.

iii) $(Y^{-1}.e)^* = (Y^*)^{-1}.e$, where e is a vector of 1's.

Therefore, all results for the ALT TPM in Section 2 are applicable to the ART TPM if the order of the sequence, {0,1,2, ,N-1,N} is reversed. There-

fore, in this section we shall concentrate on just giving the major results without details.

3.1 THE FUNDAMENTAL RECURSION

Following equations (3), (4) and (5) of Section 2.1, instead of recursive relations (6) and (7), for $j>0$ and $k\geq 0$, we have

$$a_0^{(j)} = 1$$

$$a_1^{(j)} = (1-P_{jj})/P_{j+1,j}$$

$$a_n^{(j)} = 0 \text{ for all } n<0.$$

$$a_{k+1}^{(j)} = \frac{1}{P_{j+k+1,j+k}} \left[(1-P_{j+k,j+k})\, a_k^{(j)} - \sum_{\ell=1}^{k} P_{j+k-\ell,j+k}\, a_{k-\ell}^{(j)}\right] \quad (36)$$

or alternately,

$$a_{k+1}^{(j)} = \left(\frac{1-P_{jj}}{P_{j+1,j}}\right) a_k^{(j+1)} - \sum_{\ell=1}^{k} \left[\frac{P_{j,j+\ell}}{P_{j+\ell+1,j+\ell}}\right] a_{k-\ell}^{(j+\ell+1)} \quad (37)$$

In this case we shall identify (36) as the fundamental recursion. As before we shall consider $a_k^{(j)}$'s as transition iterates.

3.2 LIMITING DISTRIBUTION OF THE MARKOV CHAIN

Theorem 3.2.1. Let $\{Q_n, n=0,1,2,\ldots\}$ be an aperiodic, irreducible finite Markov chain with an almost right triangular transition probability matrix with elements P_{ij} $(i,j=0,1,2\ldots N)$. Let $\pi = (\pi_0,\pi_1,\ldots,\pi_N)$ be the limiting distribution of the Markov chain. (a) Then the elements of π are given by

$$\pi_0 = \left[\sum_{k=0}^{N} a_k^{(0)}\right]^{-1}$$

$$\pi_j = a_j^{(0)} \Bigg/ \left(\sum_{k=0}^{N} a_k^{(0)}\right) \qquad j=1,2,\ldots,N. \quad (38)$$

(b) Further, let the first two rows of the TPM be identical (i.e., $P_{0j} = P_{1j}$ $(j=0,1,2,\ldots, N)$). Then, the elements of π are given by

$$\pi_0 = \left[\frac{a_{N-1}^{(1)}}{P_{10}} + \sum_{k=0}^{N-2} \left(\frac{a_k^{(1)}}{P_{10}} - \frac{a_k^{(2)}}{P_{21}} \right) \right]^{-1}$$

$$\pi_1 = \left(\frac{1}{P_{10}} - 1 \right) \pi_0$$

$$\pi_j = \left(\frac{a_{j-1}^{(1)}}{P_{10}} - \frac{a_{j-2}^{(2)}}{P_{21}} \right) \pi_0 \qquad j=2,3,\ldots, N. \tag{39}$$

<u>Proof</u> Consider the linear homogenous system of equations

$$\pi(I-P) = 0.$$

The elements of π are determined much the same way as in Theorem 2.2.1 of Section 2. Here we re-arrange the first N equations to express π_j $(j=1,2,\ldots N)$ in terms of π_0 and use the recursion (37) in simplifications. Part (b) of the theorem utilizes the special structure of the first two rows of the transition probability matrix P.

3.3 $(I-H)^{-1}$ and first passage times.

Analogous to lemma 2.3.4, we have

<u>Lemma 3.3.1</u>

$$x_{km} = \frac{1}{P_{j,j-1}} \left[a_{m-1}^{(j)} - \frac{P_{j,j-1}}{P_{j+k,j+k-1}} a_{m-k-1}^{(j+k)} \right]$$

$$(k=1,2,\ldots,N-1, \; m=1,2,\ldots,N)$$

$$x_{Nk} = x_{N-1,k}$$

$$x_{NN} = a_{N-1}^{(j)} \; P_{j,j-1} \tag{40}$$

Proof follows along similar lines as that of lemma 2.3.4.

<u>Corollary</u>

Let $S_k = \sum_{m=1}^{N} x_{km}$.

Then

$$S_k = S_N - \frac{1}{P_{j+k,j+k-1}} \sum_{m=0}^{N-k-1} a_m^{(j+k)} \qquad (k=1,2,\ldots,N-1)$$

$$S_N = \frac{1}{P_{j,j-1}} \left[1 + \sum_{m=1}^{N-1} a_m^{(j)} \right] \tag{41}$$

Proof follows by direct computation.

These results lead us to the mean first passage times. We have

Theorem 3.3.1. Let $\{Q_n, n=0,1,2,\ldots\}$ be an aperiodic, irreducible, finite Markov chain with an almost right triangular transition probability matrix with elements P_{ij} $(i,j=0,1,2,\ldots,N)$. *Let T_1^N be the first passage time from* state 1 to state 0 in terms of number of transitions. Then

$$E[T_1^N] = \frac{a_{N-1}^{(1)}}{P_{10}} + \sum_{m=0}^{N-2} \left(\frac{a_m^{(1)}}{P_{10}} - \frac{a_m^{(2)}}{P_{21}} \right) \tag{42}$$

Proof: We have

$$E[T_1^N] = \text{First row sum of } (I-H)^{-1}.$$

Clearly the matrix $[I-H]$ is the same as X with j replaced by 1. Hence we have

$$E[T_1^N] = S_1 = S_N - \frac{1}{P_{21}} \sum_{m=0}^{N-2} a_m^{(2)}$$

where $$S_N = \frac{1}{P_{10}} \left[1 + \sum_{m=1}^{N-1} a_m^{(1)} \right]$$

The result follows after simplification using transition iterates.

Similar results follow for $E[T_k^N(B_1)]$ when the system initially starts from state k.

Suppose we consider the first passage of the Markov chain from state $i(\geq 1)$ to the set of states $\{0,1,2,\ldots,i-1\}$. Then for $[I-H(N-i+1)]^{-1} = \| x_{km} \|$ similar methods would yield the result

$$x_{km} = \frac{1}{P_{i,i-1}} \left[a_{m-1}^{(i)} - \frac{P_{i,i-1}}{P_{i+k,i+k-1}} a_{m-k-1}^{(i+k)} \right] \qquad (1\leq k\leq N-i, 1\leq m\leq N-i+1)$$

$$x_{N-i+1,m} = x_{N-i,m} \quad (1\leq m\leq N-i) \; ; \; x_{N-i+1,N-i+1} = a_{N-i}^{(i)}/P_{i,i-1} \, .$$

$$S_k = S_{N-i+1} - \frac{1}{P_{i+k,i+k-1}} \sum_{m=0}^{N-i-k} a_m^{(i+k)} \tag{43}$$

where $$S_{N-i+1} = \frac{1}{P_{i,i-1}} \left[1 + \sum_{m=1}^{N-i} a_m^{(i)} \right]$$

Also $$E[T_i^N(B_i)] = \frac{a_{N-i}^{(i)}}{P_{i,i-1}} + \sum_{m=0}^{N-i-1} \left(\frac{a_m^{(i)}}{P_{i,i-1}} - \frac{a_m^{(i+1)}}{P_{i+1,i}} \right) . \qquad (44)$$

3.4 Remarks on computational feasibility

It may be noted that in comparison with the almost left triangular matrices, more computational effort is needed in the case of almost right triangular matrices. As can be seen from equation (38) N transition iterates are required to obtain the limiting distribution and as is evident from equation (43) a total of $2(N-i)-1$ iterates are required to determine $E[T_i^N(B_i)]$. Thus in general, the generation of these iterates would involve $N^2 + (N-i)^2 + (N-i-1)^2$ divisions/multiplications and $\frac{N(N+1)}{2} + \frac{(N-i)(N-i+1)}{2} + \frac{(N-i-1)(N-i)}{2}$ additions/subtractions. However if the first two rows of the TPM are identical it is seen from equation (39) that the vector π and $E[T_1^N]$ can be determined from the same set of $2(N-1)-1$ transition iterates. Moreover, for any i $(0<i<N)$, the determination of π together with $E[T_i^N(B_i)]$ would also require less computational effort if we take into account the identical nature of the first two rows of the transition probability matrix.

4. THE COMPUTATIONAL PROCEDURE.

In this section we present the computational procedure to determine the steady state distribution and the mean first passage times discussed earlier, in an algorithmic form suitable for computer implementation. Let $B_i = \{i, i+1, \ldots, N\}$ and for the sake of simplicity we shall denote the first passage time $T_k^N(B_i)$ by T_k^N.

Case i: P is an almost left-triangular matrix.

Initialize: Given $0 < \varepsilon \leq 1$; set $a_0^{(j)} = \varepsilon$ for $(1 \leq j \leq N + 1)$.

[Remark:

Note that the particular value of ε is very much dependent upon the magnitude of N and the computer. For small to medium-range values of N, taking $\varepsilon = 1$ may save some valuable computer time. For large values of N, however, serious overflow problems may result. This can be brought under control to some extent by suitable scaling $a_0^{(j)}$ for $j>0$.]

Step 1: Compute vector π.

Generate $a_1^{(N)}, a_2^{(N-1)}, a_3^{(N-2)}, \ldots, a_N^{(1)}$ in that order using the fundamental recursion (7). Note that these N iterates can be successively generated by setting k=0, j=N-k and successively incrementing k up to N-1.

Set $\pi_j = \left(\frac{1}{\varepsilon}\right)\left(a_{N-j}^{(j+1)} \Big/ p_{j,j+1}\right)$ $(0 \leq j \leq N - 1)$

Normalization:

Set $\pi_N = [1 + \sum_{j=0}^{N-1} \pi_j]^{-1}$

and $\pi_j = \pi_j * \pi_N$ $(0 \leq j \leq N - 1)$

Go to Step 2 or 4.

Step 2: Compute the expected values of the function T_k^N $(i \leq k \leq N)$.

[Remark: Note that $E[T_K^N]$ is given by the $(k-i+1)^{th}$ component of the column

vector of row sums of the fundamental matrix. If we only need $E[T_i^N]$ $(0 < i < N)$, we can obtain this from the above N iterates using the following expression.

$$E[T_i^N] = \left(\frac{1}{\varepsilon}\right)\left(\frac{1}{a_{N-i+1}^{(i)}}\right)\left[1 + \sum_{j=1}^{N-i} \frac{a_{N-i+1-j}^{(i+j)}}{p_{i+j-1,i+j}}\right] \tag{45}$$

Otherwise, in addition to these N iterates, we need to generate $\frac{1}{2}(N-i)^2$ iterates to compute $E[T_k^N]$ for all $k \in B_i$.]

Obtain the value of $E[T_i^N]$ using equation (45). For any fixed $k > i$, generate $a_1^{(k-1)}$, $a_2^{(k-2)}$, $a_3^{(k-3)}$, ..., $a_{k-i}^{(i)}$ in that order using recursion (7).

$$\text{Set } E[T_K^N] = E[T_i^N]\, a_{k-i}^{(i)} - \frac{1}{\varepsilon} \sum_{j=1}^{k-i} \frac{a_{k-i-j}^{(i+j)}}{p_{i+j-1,i+j}}$$

Go to Step 4.

Step 4: Stop.

Case ii: P is an almost right triangular matrix.

Initialize: Given $0 < \varepsilon \leq 1$; set $a_0^{(j)} = \varepsilon$ for $(0 \leq j \leq N)$

Step 1: Compute vector π.

If the first two rows of P are identical and if one is interested in obtaining $E[T_1^N]$ subsequently then go to Minimize.

Generate $a_k^{(0)}$ $(1 \leq k \leq N)$ successively using recursion (36)

Set $\pi_j = \frac{1}{\varepsilon} a_j^{(0)}$ $(1 \leq j \leq N)$

Go to Normalize.

Minimize:

Generate $a_k^{(t)}$ $(1 \leq k \leq N - t)$ for t=1, 2 using (36)

Set $\pi_1 = \left(\frac{1-P_{10}}{P_{10}}\right)$

$$\pi_j = \left(\frac{1}{\varepsilon}\right)\left[\frac{a_{j-1}^{(1)}}{P_{10}} - \frac{a_{j-2}^{(2)}}{P_{21}}\right] \quad (2 \le j \le N)$$

Normalize:

Set $\pi_0 = \left[1 + \sum_{j=1}^{N} \pi_j\right]^{-1}$

and

$$\pi_j = \pi_j * \pi_0 \quad (1 \le j \le N)$$

Go to step 2 or 4.

Step 2: Compute the expected values of the function T_k^N $(i \le k \le N)$

We need to generate $\frac{1}{2}(N-i)^2$ iterates to compute $E[T_k^N]$ for all $k \varepsilon B_i$.

Generate $a_m^{(i)}$ $(1 \le m \le N - i)$ successively using (36)

Set

$$E[T_N^N] = \left(\frac{1}{\varepsilon}\right)\left(\frac{1}{P_{i,i-1}}\right)\sum_{m=0}^{N-i} a_m^{(i)}.$$

For any fixed $k \varepsilon B_i$, generate $a_m^{(k+1)}$ $(1 \le m \le N - k - 1)$

using (36)

Set

$$E[T_k^N] = E[T_N^N] - \left(\frac{1}{\varepsilon}\right)\left(\frac{1}{P_{k+1,k}}\right)\sum_{m=0}^{N-k-1} a_m^{(k+1)}$$

Go to step 4.

Step 4: Stop.

5. THE QUEUE G/M/s/N - AN EXAMPLE

Consider an s-channel queueing system with two classes of customers with the same service rate and exponential service time distribution. Let t_1, t_2, ... be the arrival epochs in the system, and $Z_n=t_n-t_{n-1}$ be independent and identically distributed random variables with

$$P[Z_n \leq x] = A(x) \quad (x \geq 0)$$

and $E[Z_n] = \lambda^{-1}$, n=1,2,3... . When an arrival occurs, the probability that it is a class 1 arrival is p and the probability that it is a class 2 arrival is 1-p.

The customers are served by s service channels in parallel. All services are assumed to be exponential with the same mean μ^{-1}. The buffer capacity is such that $N(\geq s)$ customers are allowed in the system at any time. Class 1 customers have a higher priority in gaining access to the system. If the number of customers in the system is $\geq k$ $(k \leq s)$, an arriving class 2 customer is denied access to any service channel, and immediately he leaves the system without service. But, a class 1 customer is denied access to the system only when the number of customers in the system is N (no buffer space available). Class 1 customers waiting in the buffer are served on a 'first in first out' basis. Let Q(t) be the number of customers in the system at time t and Q_n be the number of customers in the system just before the nth arrival epoch. (Note that the incoming customer is not included in Q_n.) Becuase of the exponential nature of service times, it is well known that $\{Q_n, n=1,2,3,... \}$ is a Markov chain imbedded in the stochastic process $\{Q(t), t\geq 0\}$. A Markov chain is completely specified by its transition probability matrix. Let

$$P_{ij} = P[Q_{n+1} = j \mid Q_n = i] \quad (i,j = 0,1,2,\ldots, N).$$

We have

$$P_{ij} = \binom{i+1}{j} \int_0^\infty (1-e^{-\mu x})^{i+1-j} e^{-j\mu x} dA(x) \qquad i < k,\ j \le i+1 \tag{45a}$$

$$= p\binom{i+1}{j} \int_0^\infty (1-e^{-\mu x})^{i+1-j} e^{-j\mu x} dA(x)$$

$$+ (1-p)\binom{i}{j} \int_0^\infty (1-e^{-\mu x})^{i-j} e^{-j\mu x} dA(x) \qquad k \le i < s,\ j \le i+1 \tag{45b}$$

$$= p \int_0^\infty e^{-s\mu x} \frac{(s\mu x)^{i+1-j}}{(i+1-j)!} dA(x)$$

$$+ (1-p) \int_0^\infty e^{-s\mu x} \frac{(s\mu x)^{i-j}}{(i-j)!} dA(x) \qquad i \le s-1,\ i+1 \ge j \ge s \tag{45c}$$

$$= p \int_{x=0}^\infty \int_{t=0}^x e^{-s\mu t} \frac{(s\mu t)^{i-s}}{(i-s)!}\, s\mu \binom{s}{j} [1-e^{-\mu(x-t)}]^{s-j} e^{-j\mu(x-t)} dt\, dA(x)$$

$$+ (1-p) \int_{x=0}^\infty \int_{t=0}^x e^{-s\mu t} \frac{(s\mu t)^{i-1-s}}{(i-1-s)!}\, s\mu \binom{s}{j} [1-e^{-\mu(x-t)}]^{s-j}$$

$$e^{-j\mu(x-t)} dt\, dA(x) \qquad i \ge s,\ j < s. \tag{45d}$$

$$= \int_0^\infty e^{-s\mu x} \frac{(s\mu x)^{N-j}}{(N-j)!} dA(x) \qquad i = N,\ N \ge j \ge s \tag{45e}$$

$$= \int_{x=0}^\infty \int_{t=0}^x e^{-s\mu t} \frac{(s\mu t)^{N-1-s}}{(N-1-s)!}\, s\mu \binom{s}{j} [1-e^{-\mu(x-t)}]^{s-j}$$

$$e^{-j\mu(x-t)} dt\, dA(x) \qquad i = N,\ j < s\ . \tag{45f}$$

For computational purposes the above expressions can be simplified by expressing them in terms of the following transform:

$$\gamma_j(\delta) = \int_0^\infty e^{-\delta x} \frac{(\delta x)^j}{j!} dA(x) \qquad (j = 0,1,2,\ldots\ ) \tag{46}$$

Then we get

$$P_{ij} = \binom{i+1}{j} \sum_{r=0}^{i-j+1} (-1)^r \binom{i-j+1}{r} \gamma_0((r+j)\mu) \qquad i < k,\ i-j+1 \qquad (47a)$$

$$= p\binom{i+1}{j} \sum_{r=0}^{i-j+1} (-1)^r \binom{i-j+1}{r} \gamma_0((r+j)\mu)$$

$$+ (1-p)\binom{i}{j} \sum_{r=0}^{i-j} (-1)^r \binom{i-j}{r} \gamma_0((r+j)\mu) \qquad k \le i < s,\ j \le i+1 \qquad (47b)$$

$$= p\,\gamma_{i-j+1}(s\mu) + (1-p)\,\gamma_{i-j}(s\mu) \qquad i \le s-1,\ i+1 \ge j \ge s \qquad (47c)$$

$$= p\binom{s}{j} \sum_{\ell=i-s+1}^{\infty} \left[\sum_{r=0}^{s-j} \binom{s-j}{r} \left(\frac{s-j-r}{s}\right)^{\ell-i+s-1}\right] \gamma_\ell(s\mu)$$

$$+ (1-p)\binom{s}{j} \sum_{\ell=i-s}^{\infty} \left[\sum_{r=0}^{s-j} \binom{s-j}{r} \left(\frac{s-j-r}{s}\right)^{\ell-i+s}\right] \gamma_\ell(s\mu) \qquad i \ge s,\ j < s \qquad (47d)$$

$$= \gamma_{N-j}(s\mu) \qquad i = N,\ N \ge j \ge s$$

$$= \binom{s}{j} \sum_{\ell=N-s}^{\infty} \left[\sum_{r=0}^{s-j} (-1)^r \binom{s-j}{r} \left(\frac{s-j-r}{s}\right)^{\ell-N+s}\right] \gamma_\ell(s\mu) \qquad i = N,\ j < s. \qquad (47f)$$

Measures of performance

ρ : Offered load per channel ($= \lambda/s\mu$)

ρ^* : Effective offered load per channel (channel utilization)

$= \rho\,[a_1 p + a_2(1-p)]$

where $a_1 = 1-\pi_N$; $a_2 = 1 - \sum_{r=k}^{N} \pi_r$

$E(Q)$: Mean number in system at arrival epochs.

$V(Q)$: Variance of number in system at arrival epochs.

$E(W)$: Mean delay for class 1 customers

$$= \frac{p}{s\mu[1-\pi_N]} \sum_{j=1}^{N-s-1} (j+1)\, \pi_{s+j} \, .$$

PB_i : Probability of blocking for class i (i=1,2)

$$PB_1 = p\pi_N \; ; \; PB_2 = (1-p) \sum_{r=k}^{N} \pi_r$$

S_A : System adequacy (system response to offered load)

$$= \rho^* / \rho$$

AR_1 : Access ratio for class 1

$$= \frac{a_1}{a_1 + a_2}$$

E(BC) : Mean busy cycle

$$= \frac{1}{\lambda} \, [1 + \gamma_0(\mu) \; E[T_1^N \, (B_1)]].$$

System parameters	Variable parameter	ρ	$\rho^*=C_u$	$E_{ARR}(Q)$	$V_{ARR}(Q)$	E (W)	PB_2^*	S_A	AR_1	E(BC)
↑ N=15	1.0	.125	.125	.693	.553	.000	.001	.999	.500	2.18
s=8 k=4	3.0	.375	.337	2.381	1.309	.000	.101	.899	.546	7.74
p=.4 λ variable	5.0	.625	.451	3.351	1.341	.000	.278	.722	.651	34.36
m=4	7.0	.875	.524	3.981	1.521	.001	.401	.600	.750	119.27
μ=1 ↓	9.0	1.125	.588	4.535	1.990	.004	.478	.523	.830	345.67
↑ N=18 s=10	10	↑	.388	3.468	1.906	.003	.028	.971	.519	29.02
k=6 p=.6	6		.387	3.489	1.996	.003	.095	.968	.521	26.26
λ=4 m variable	4	.4	.386	3.515	2.108	.003	.100	.965	.523	23.49
μ=1	2		.381	3.591	2.428	.003	.112	.954	.531	18.03
↓	1	↓	.374	3.737	3.026	.004	.132	.934	.545	12.82
↑ N variable s=12	20	↑	.568	6.545	2.761	.076	.267	.852	.586	1318.50
k=8 p=.5	18		.568	6.545	2.761	.077	.267	.852	.586	1318.50
λ=8 m=3	16	.667	.568	6.545	2.760	.077	.267	.852	.586	1318.50
	14		.568	6.545	2.759	.077	.267	.852	.586	1318.46
μ=1 ↓	13	↓	.568	6.544	2.754	.077	.267	.852	.586	1318.32

*PB_1 was found .000 in all cases.

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Imbedded Markov chains of finite queueing systems with unit jumps at regeneration points have an almost left triangular (in systems of the type G/M/s/N - in Kendall notation modified to include system capacity) or an almost right triangular (in systems of the type M/G/1/N) structure. Using this structure a fundamental recursion on the elements of the transition probability matrix is developed which in turn helps derive first passage as well as equilibrium results in computationally feasible forms. The computational procedure is illustrated using the system G/M/s/N with two arrival ~~classes and priority service.~~

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